The Wirtinger Presentation

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The study of knots is an active area of modern mathematical research. Much of their study concerns the creation and computation of various *knot invariants*—algebraic objects associated to each knot which can be used to distinguish non-equivalent knots from each other. Many such invariants exist, but a well known one is the fundamental group of the complement of a knot, otherwise called the *knot group*. In this paper we present a computation of this invariant for a knot embedded in 3-space.

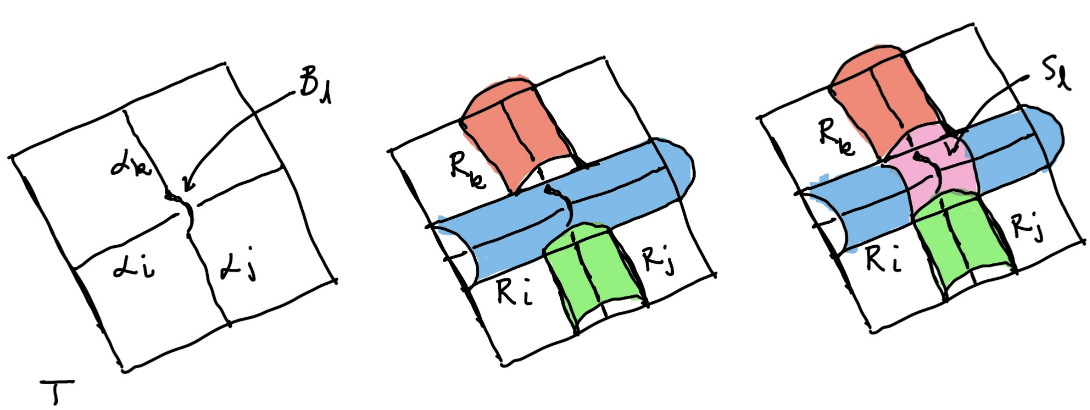
The *Wirtinger presentation* is a finite group presentation of the fundamental group of the complement of a knot in 3-space. First, we write down some definitions

Definition. *A **knot** is an embedding of S^1 in R^3.*

Definition. *The **fundamental group** of X based at x_0, $\pi_1(X, x_0)$, is the set of equivalence classes of loops based at x_0 with binary operation $[\alpha][\gamma] = [\alpha \cdot \gamma]$.*

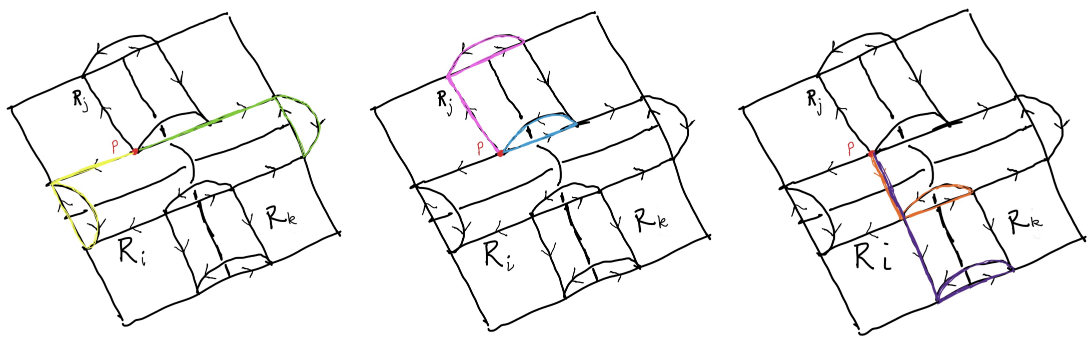
Definition. *A **CW-complex** is any space X which can be constructed by starting off with a discrete collection of points called 0-cells that make up the 0-skeleton X^0, then attaching 1-cells e^1_α to X^0 along their boundaries S^0, writing X^1 for the 1-skeleton obtained by attaching the 1-cells to X^0, then attaching 2-cells e^2_α to X^1 along their boundaries S^1, writing X^2 for the 2-skeleton, and so on, giving spaces X^n for every n. A CW-complex is any space that has this sort of decomposition into subspaces X^n built up in such a way that the X^ns exhaust all of X. In particular, X^n may be built from X^{n-1} by attaching infinitely many n-cells e^n_α, and the attaching maps $S^{n-1} \to X^{n-1}$ may be any continuous maps.*

Definition. *A **deformation retract** of a space X onto a subspace A is a family of maps $f_t : X \to X, t \in I$, such that $f_0 = id$ (the identity map), $f_1(X) = A$, and $f_t|A = id$ for all t. The family f_t should be continuous in the sense that the associated map $X \times I \to X, (x,t) \mapsto f_t(x)$ is continuous.*

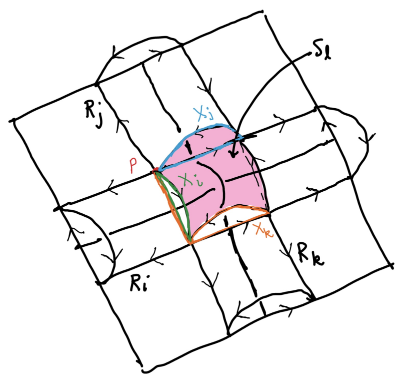


To begin, let K be a smooth or piecewise linear knot in $\mathbb{R}^3$. Position the knot to lie almost flat, so that K consists of finitely many disjoint arcs α_i and finitely many disjoint arcs B_l where K crosses over itself as shown in the first figure above. Now, we build a 2-dimensional CW-complex X that is a deformation retract of $\mathbb{R}^3 - K$. First, just above each arc α_i place a long, thin rectangular strip R_i, curved to run parallel to α_i along the full length of α_i and arched so that the two long edges of R_i are identified with points of the rectangle T. This is shown in the second figure. Any arcs B_l that cross over α_i are positioned to lie in R_i. Then, over each are B_l put a square S_l, bent downward along its four edges so that these edges are identified with points of three strips R_i, R_j, and R_k as in the third figure. The knot K is now a subspace of X, but after we lift K up slightly into the complement of X, it becomes a deformation retract of $\mathbb{R}^3 - K$.

Theorem. *The fundamental group of $\mathbb{R}^3 - K$, $\pi_1(\mathbb{R}^3 - K)$, has a group presentation with one generator x_i for each strip R_i and one relation of the form $x_i x_j x_i^{-1} = x_k$ for each square S_l.*



Notice that the loops based at p at opposite ends of the strips are homotopy equivalent. Intuitively, the yellow loop can continuously deform into the green loop via the strip R_i. Similarly, the pink loop can continuously deform into the blue loop via R_j and the purple loop can continuously deform into the orange loop via R_k.



After attaching the square S_l, we see that the loop x_k is homotopy equivalent to the loop $x_i x_j x_i^{-1}$ via S_l.

In the proof of the Theorem we will use the following Proposition and figures:

Proposition 1. (Proposition 1.26 in [1]) *Suppose we attach a collection of 2-cells e^2_α to a path-connected space X via maps $\varphi_\alpha : S^1 \to X$, producing a space Y. If s_0 is a basepoint of S^1 then φ_α determines a loop based at $\varphi_\alpha(s_0)$ called φ_α. For different α's the basepoints of the loops φ_α may vary. Choose a basepoint $x_0 \in X$ and a path γ_α in X from x_0 to $\varphi_\alpha(s_0)$ for each α. Then $\gamma_\alpha \varphi_\alpha \gamma_\alpha^{-1}$ is a loop at x_0 where $\gamma_\alpha^{-1}(t) = \gamma_\alpha(1-t)$ is the path inverse. The inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is the normal subgroup N generated by all the loops $\gamma_\alpha \varphi_\alpha \gamma_\alpha^{-1}$ for varying α. Thus, $\pi_1(Y) \cong \pi_1(X)/N$.*

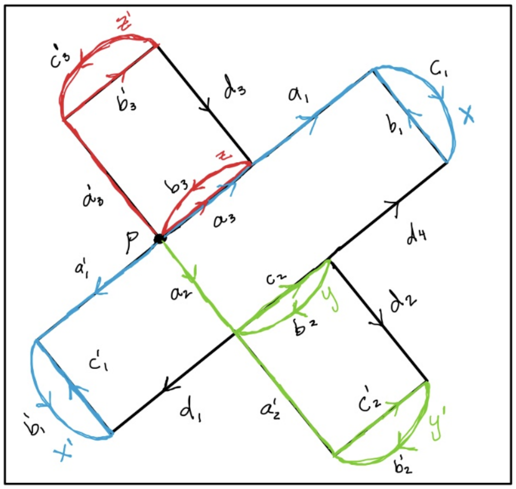


Figure A

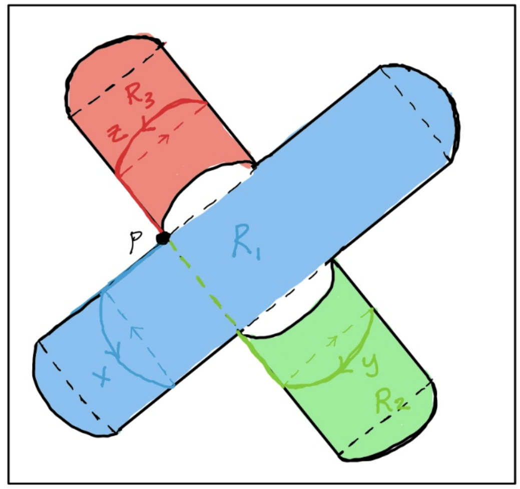


Figure B

Proof. It is sufficient to prove $\pi_1(X,p) \cong \langle x_1, x_2, x_3 \mid x_1x_2x_1^{-1} = x_3\rangle$. Let Z be the one-dimensional space in Figure A. Notice that $\pi_1(Z,p) \cong \langle x, y, z, x', y', z'\rangle$ where the loops are defined as

$$x = a_3a_1c_1b_1a_1^{-1}a_3^{-1}$$

$$y = a_2c_2b_2a_2^{-1}$$

$$z = a_3b_3$$

$$x' = a_1'b_1'c_1'a_1'^{-1}$$

$$y' = a_2a_2'c_2'b_2'a_2'^{-1}a_2^{-1}$$

$$z' = a_3'b_3'c_3'a_3'^{-1}.$$

Next, attach the 2-cells R_1, R_2, and R_3 to Z in the following way: attach R_1 via the homomorphism $\varphi_1 : S^1 \to \mathbb{Z}$ where $\varphi_1 = a_1'b_1'd_1c_2d_4^{-1}c_1^{-1}a_1^{-1}a_3^{-1}$, R_2 via the homomorphism $\varphi_2 : S^1 \to \mathbb{Z}$ where $\varphi_2 = a_2'b_2'^{-1}d_2b_2$, and R_3 via the homomorphism $\varphi_3 : S^1 \to \mathbb{Z}$ where $\varphi_3 = b_3^{-1}d_3^{-1}c_3'a_3'^{-1}$. Now we have the space Y shown in Figure B. By Proposition 1, $\pi_1(Y,p) \cong \pi_1(Z,p)/N$ where $N = \langle \varphi_1, a_2\varphi_2a_2^{-1}, \varphi_3\rangle$. Let $r_1 = a_1'c_1'^{-1}d_1c_2d_4^{-1}b_1a_1^{-1}a_3^{-1}$, $r_2 = a_2a_2'c_2'd_2c_2^{-1}a_2^{-1}$, and $r_3 = a_3d_3^{-1}b_3'^{-1}a_3'^{-1}$ such that $x'r_1x^{-1} = (a_1'b_1'c_1'a_1'^{-1})(a_1'c_1'^{-1}d_1c_2d_4^{-1}b_1a_1^{-1}a_3^{-1})(a_3a_1b_1^{-1}c_1^{-1}a_1^{-1}a_3^{-1}) = a_1'b_1'd_1c_2d_4^{-1}c_1^{-1}a_1^{-1}a_3^{-1} = \varphi_1$, $y'^{-1}r_2y = (a_2a_2'b_2'^{-1}c_2'^{-1}a_2'^{-1}a_2^{-1})(a_2a_2'c_2'd_2c_2^{-1}a_2^{-1})(a_2c_2b_2a_2^{-1}) = a_2a_2'b_2'^{-1}d_2b_2a_2^{-1} = a_2\varphi_2a_2^{-1}$, and $z^{-1}r_3z' = (b_3^{-1}a_3^{-1})(a_3d_3^{-1}b_3'^{-1}a_3'^{-1})(a_3'b_3'c_3'a_3'^{-1}) = \varphi_3$. Since r_1, r_2, r_3 are homotopic to the constant path at p, we get that $\varphi_1 \simeq x'x^{-1}$, $a_2\varphi_2a_2^{-1} \simeq y'^{-1}y$, and $\varphi_3 \simeq z^{-1}z'$. Therefore, $\pi_1(Y,p) \cong \langle x, y, z, x', y', z' \mid x'x^{-1}, y'^{-1}y, z^{-1}z'\rangle \cong \langle xN, yN, zN\rangle \cong \langle x, y, z\rangle$. The loops x, y, z are shown in Figure B.

Now, we contruct the space X by adding a 2-cell S_1 to the space Y via the homomorphism $\phi_1 = e_1e_2e_3e_4$ where e_1, e_2, e_3, e_4 are the edges of the square S_1 orientated counterclockwise starting at p. Then $\phi_1 \simeq xy^{-1}x^{-1}z$. By Proposition 1, $\pi_1(X,p) \cong \pi_1(Y,p)/M$ where $M = \langle xy^{-1}x^{-1}z\rangle$. Thus, $\pi_1(X,p) \cong \langle x, y, z \mid xy^{-1}x^{-1}z\rangle$. Finally, note that $\langle xy^{-1}x^{-1}z\rangle = \langle z^{-1}xyx^{-1}\rangle$ since $xy^{-1}x^{-1}z$ and $z^{-1}xyx^{-1}$ are inverses and let $x = x_1, y = x_2$, and $x = x_3$. Then we get

$$\pi_1(X,p) \cong \langle x_1, x_2, x_3 \mid x_3^{-1}x_1x_2x_1^{-1}\rangle \cong \langle x_1, x_2, x_3 \mid x_1x_2x_1^{-1} = x_3\rangle.$$

□

References

[1] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2001.